Noname manuscript No.
(will be inserted by the editor)

Spectral characterization of families of split graphs

Milica Anđelić · Domingos M. Cardoso

Published in Graphs and Combinatorics (2015) 31: 59-72, DOI 10.1007/s00373-013-1387-8
Received: November, 2011 / Accepted: March 2013.

Abstract An upper bound for the sum of the squares of the entries of the principal eigenvector corresponding to a vertex subset inducing a k-regular subgraph is introduced and applied to the determination of an upper bound on the order of such induced subgraphs. Furthermore, for some connected graphs we establish a lower bound for the sum of squares of the entries of the principal eigenvector corresponding to the vertices of an independent set. Moreover, a spectral characterization of families of split graphs, involving its index and the entries of the principal eigenvector corresponding to the vertices of the maximum independent set is given. In particular, the complete split graph case is highlighted.

Keywords split graph · largest eigenvalue · principal eigenvector · programming involving graphs

Mathematics Subject Classification (2000) MSC 05C50 · MSC 90C35

Research supported in part by *FEDER* funds through *COMPETE* - Operational Programme Factors of Competitiveness ("Programa Operacional Factores de Competitividade") and by Portuguese funds through the Center for Research and Development in Mathematics and Applications (University of Aveiro) and the Portuguese Foundation for Science and Technology ("FCT - Fundação para a Ciência e a Tecnologia"), within project PEst-C/MAT/UI4106/2011 with COMPETE number FCOMP-01-0124-FEDER-022690. The work of first author was supported by Serbian Ministry of Science, Project 174033.

Milica Anđelić
Center for Research and Development in Mathematics and Applications, Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal; Faculty of Mathematics, University of Belgrade, Serbia
E-mail: milica.andelic@ua.pt

Domingos M. Cardoso
Center for Research and Development in Mathematics and Applications, Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal
E-mail: dcardoso@ua.pt

1 Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order and the size of G are denoted by $\nu(= |V|)$ and $\epsilon(= |E|)$, respectively. We write $u \sim v$ whenever the vertices u and v are adjacent, and A_G stands for the $(0, 1)$-adjacency matrix of G. The neighborhood of a vertex $i \in V(G)$, that is, the set of vertices adjacent to i, is denoted by $N_G(i)$, the degree of i is $d_i = |N_G(i)|$, $\Delta(G) = \max_{i\in V(G)} d_i$ and $\delta(G) = \min_{i\in V(G)} d_i$. The subgraph of G induced by the vertex subset $T \subset V(G)$ is denoted by $G[T]$. On the other hand, the set of edges with just one end vertex in T is denoted $\partial(T)$. The largest eigenvalue of A_G, denoted by $\lambda_1(G)$, is commonly called the *index* or *spectral radius* of G. If $T \subset V(G)$ is a vertex subset of G its complement is denoted $\bar{T}$ (that is, $\bar{T} = V(G) \setminus T$). A stable or independent set (clique) is a subset of pairwise non-adjacent (adjacent) vertices. The *stability number* or *independence number* (*clique number*) of a graph G, denoted by $\alpha(G)$ ($\omega(G)$) is the cardinality of a stable set (clique) with maximum cardinality. For further details the reader is referred to [5,6].

Let us start recalling that a *split graph* is a graph whose vertex set can be divided into two subsets, one being a co-clique, the other being a clique, and all other edges (the cross-edges) join two vertices belonging to different subsets. If each vertex in co-clique is adjacent to all vertices in clique then G is called a *complete split graph.*

Since, in the case of connected graphs, A_G is a nonnegative and irreducible matrix, the eigenvector corresponding to the index can be taken to be positive. Unless stated otherwise, we will denote such vector by

$$\mathbf{x} = (x_1, x_2, \ldots, x_\nu)^T,$$

and assume that $\sum_{i=1}^{\nu} x_i^2 = 1$, i.e., $\mathbf{x}$ is a unit vector known as the *principal eigenvector* of G [6, p.16]. The results of this note are stated in terms of this eigenvector.

In [3] the following result was shown.

Theorem 1 *[3] Let G be a connected graph. If $S \subset V(G)$ is an independent set, we have*

$$\sum_{i\in S} x_i^2 \leq \frac{1}{2}.$$

Moreover, G is bipartite with S as one color class if and only if $\sum_{i\in S} x_i^2 = \frac{1}{2}$.

However, there are bipartite graphs G with color classes V_1 and V_2 such that none of them are maximum independent sets, that is, there exists an independent set $S \subset V(G)$ such that $|S| > \max\{|V_1|, |V_2|\}$. As consequence, even for bipartite graphs G (as it is the case of the graph depicted in the next figure), there are maximum independent sets $S \subset V(G)$ such that

$$\sum_{i\in S} x_i^2 < \frac{1}{2}.$$

In the next section the above result is extended to the case of vertex subsets inducing k-regular subgraphs, with $k \in \mathbb{N} \cup \{0\}$. This extension is applied to the

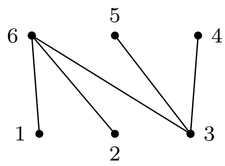

Fig. 1 A bipartite graph where none of the color classes $V_1 = \{1, 2, 3\}$ and $V_2 = \{4, 5, 6\}$ is a maximum independent set and the maximum independent set $S = \{1, 2, 4, 5\}$ is such that $\sum_{i \in S} x_i^2 < 1/2$.

determination of an upper bound on the order of k-regular induced subgraphs. A convex quadratic upper bound on the order of k-regular induced subgraphs was obtained in [1] (see also [2]). In Section 3, for some connected graphs, a lower bound for the sum of squares of the entries of the principal eigenvector corresponding to the vertices of an independent set is introduced. Based on the previous result, in Section 4, families of split graphs are characterized by a function of its index and the entries of the principal eigenvector corresponding to the vertices of the maximum independent set. Furthermore, the complete split graph case is highlighted. In Section 5, numerical examples are presented.

2 An upper bound on the sum of squares of the entries of the principal eigenvector corresponding to a vertex subset inducing a *k*-regular subgraph

Now, we introduce the following generalization of Theorem 1.

Theorem 2 *Let G be a connected graph, such that its index is $\lambda_1 = \lambda_1(G)$. If $S \subset V(G)$ induces a k-regular subgraph, with $k \in \mathbb{N} \cup \{0\}$, then*

$$\sum_{i \in S} x_i^2 \leq \frac{\lambda_1}{2\lambda_1 - k}. \tag{1}$$

Furthermore, (1) *holds as equality if and only if $\bar{S} = V(G) \backslash S$ is an independent set and one of the following conditions holds: (i) $k = 0$ (and then G is bipartite) or (ii) $x_i =$ constant for all the vertices i of each component of $G[S]$.*

Proof Let $A_G = \begin{pmatrix} A_{G[S]} & B \\ B^T & A_{G[\bar{S}]} \end{pmatrix}$, λ_1 the index of G and $\mathbf{x}$ its corresponding principal eigenvector as introduced above, such that $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, where the entries of $x = (x_1, \ldots, x_m)^T$ correspond to the vertices in S and the entries in $y = (y_1, \ldots, y_n)^T$ correspond to the vertices in $\bar{S}$. Since $\lambda_1 x_i = \sum_{j \in N_G(i) \cap S} x_j + \sum_{j \in N_G(i) \cap \bar{S}} y_j$, then $\lambda_1 x_i^2 = \sum_{j \in N_G(i) \cap S} x_i x_j + \sum_{j \in N_G(i) \cap \bar{S}} x_i y_j$. Therefore, it follows

$$\begin{aligned}
\lambda_1 \sum_{i \in S} x_i^2 &= 2 \sum_{ij \in E(G[S])} x_i x_j + \sum_{ij \in \partial(S)} x_i y_j \\
&= \frac{x^T A_{G[S]} x}{2} + \sum_{ij \in E(G[S])} x_i x_j + \sum_{ij \in \partial(S)} x_i y_j \\
&= \frac{x^T A_{G[S]} x}{2} + \frac{\mathbf{x}^T A_G \mathbf{x}}{2} - \frac{1}{2} y^T A_{G[\bar{S}]} y \\
&= \frac{\lambda_1}{2} + \frac{1}{2}\left(x^T A_{G[S]} x - y^T A_{G[\bar{S}]} y\right). \qquad (2)
\end{aligned}$$

Since the index of $G[S]$ is $\lambda_1(G[S]) = k$, it follows that $x^T A_{G[S]} x \leq k\|x\|^2 = k \sum_{i \in S} x_i^2$, and then

$$\sum_{i \in S} x_i^2 \leq \frac{\lambda_1}{2\lambda_1} + \frac{k \sum_{i \in S} x_i^2}{2\lambda_1} - \frac{y^T A_{G[\bar{S}]} y}{2\lambda_1}$$

$$\Updownarrow$$

$$\sum_{i \in S} x_i^2 (1 - \frac{k}{2\lambda_1}) \leq \frac{\lambda_1}{2\lambda_1} - \frac{y^T A_{G[\bar{S}]} y}{2\lambda_1}$$

$$\Updownarrow$$

$$\sum_{i \in S} x_i^2 \leq \frac{\lambda_1}{2\lambda_1 - k} - \frac{1}{2\lambda_1 - k} y^T A_{G[\bar{S}]} y. \qquad (3)$$

Therefore, note that $y^T A_{G[\bar{S}]} y \geq 0$, we obtain $\sum_{i \in S} x_i^2 \leq \frac{\lambda_1}{2\lambda_1 - k}$.
Let us prove the second part of the theorem.

1. First, assume that $\bar{S}$ is an independent set (then $y^T A_{G[\bar{S}]} y = 0$) and one of the conditions (i) or (ii) holds.
   (i) If $k = 0$, then S is an independent set and therefore G is bipartite with S as one of its two color classes. Applying Theorem 1, the inequality (1) holds as equality.
   (ii) If $x_i =$ constant for every vertex i in each component of $G[S]$, then x is an eigenvector of $A_{G[S]}$ corresponding to the eigenvalue k. From (2), $\sum_{i \in S} x_i^2 = \frac{\lambda_1}{2\lambda_1} + \frac{1}{2\lambda_1} x^T A_{G[S]} x \Leftrightarrow x^T A_{G[S]} x = 2\lambda_1 \sum_{i \in S} x_i^2 - \lambda_1$. Therefore,
   $$k = \frac{x^T A_{G[S]} x}{\sum_{i \in S} x_i^2} = 2\lambda_1 - \frac{\lambda_1}{\sum_{i \in S} x_i^2} \Rightarrow \sum_{i \in S} x_i^2 = \frac{\lambda_1}{2\lambda_1 - k}.$$
2. Conversely, suppose that the inequality (1) holds as equality. Then, from (3), $y^T A_{G[\bar{S}]} y = 0$ and, since the entries of y are all positive, we may conclude that $\bar{S}$ is an independent set. On the other hand, from (2) and taking into account the equality $\sum_{i \in S} x_i^2 = \frac{\lambda_1}{2\lambda_1 - k}$,
   $$\begin{aligned}
   \sum_{i \in S} x_i^2 = \frac{\lambda_1}{2\lambda_1} + \frac{1}{2\lambda_1} x^T A_{G[S]} x &\Leftrightarrow 1 = \frac{2\lambda_1 - k}{2\lambda_1} + \frac{1}{2\lambda_1} \frac{x^T A_{G[S]} x}{\sum_{i \in S} x_i^2} \\
   &\Leftrightarrow \frac{x^T A_{G[S]} x}{\|x\|^2} = k.
   \end{aligned}$$

Therefore, x is an eigenvector corresponding to the eigenvalue k (the regularity of $G[S]$) and thus or $k = 0$ or $x_i =$ constant for every vertex i in each component of $G[S]$.

□

From this theorem, we have the following corollary.

Corollary 1 *If G is a connected p-regular graph of order ν and $S \subset V(G)$ induces a k-regular subgraph, then $|S| \le \nu \frac{p}{2p-k}$.*

Proof Assume that G is p-regular (then its index is p and the corresponding principal eigenvector has entries $x_i = \frac{1}{\sqrt{\nu}}, i = 1, \ldots, \nu$) and $S \subset V(G)$ induces a k-regular subgraph. Then, applying Theorem 2, it follows

$$\sum_{i \in S} x_i^2 = \frac{|S|}{\nu} \le \frac{\lambda_1}{2\lambda_1 - k} = \frac{p}{2p-k}.$$

Therefore, $|S| \le \nu \frac{p}{2p-k}$. □

As immediate consequence, if G is a connected regular graph of order ν, then $\alpha(G) \le \frac{\nu}{2}$.

From now on, let us assume that $\underline{x}$ and $\bar{x}$ denote the minimum and maximum of the entries of the principal eigenvector $\mathbf{x}$ of the connected graph G. As a consequence of Theorem 2, we deduce the following corollary.

Corollary 2 *Let G be a connected graph of order ν, with index λ_1 and principal eigenvector $\mathbf{x}$. If $S \subset V(G)$ induces a k-regular subgraph with $k \in \mathbb{N} \cup \{0\}$, then*

$$|S| \le \min\{\lfloor \frac{\lambda_1}{\underline{x}^2(2\lambda_1 - k)} \rfloor, \lfloor \nu - \frac{\lambda_1 - k}{\bar{x}^2(2\lambda_1 - k)} \rfloor\}. \tag{4}$$

Proof Let us suppose that $S \subset V(G)$ induces a k-regular subgraph of G. From Theorem 2, $\sum_{i \in S} x_i^2 \le \frac{\lambda_1}{2\lambda_1 - k}$ and then $\sum_{j \in \bar{S}} {y_j}^2 \ge 1 - \frac{\lambda_1}{2\lambda_1 - k}$. Therefore,

$$|S|\underline{x}^2 \le \sum_{i \in S} x_i^2 \le \frac{\lambda_1}{2\lambda_1 - k} \Rightarrow |S| \le \frac{\lambda_1}{\underline{x}^2(2\lambda_1 - k)}$$

$$(\nu - |S|)\bar{x}^2 \ge \sum_{j \in \bar{S}} {y_j}^2 \ge \frac{\lambda_1 - k}{2\lambda_1 - k} \Rightarrow \nu - \frac{\lambda_1 - k}{\bar{x}^2(2\lambda_1 - k)} \ge |S|.$$

3 A lower bound on the sum of squares of the entries of the principal eigenvector corresponding to the vertices of an independent set

Throughout this section, we consider a connected graph G with a vertex subset $S \subset V(G)$, such that $A_G = \begin{pmatrix} A_{G[S]} & B \\ B^T & A_{G[\bar{S}]} \end{pmatrix}$. Then

$$\lambda_1 = x^T A_{G[S]} x + 2\, x^T B y + y^T A_{G[\bar{S}]} y, \tag{5}$$

where $x = (x_1, x_2, \ldots, x_m)^T$ is such that x_j is the coordinate of $\mathbf{x}$ corresponding to the vertex $j \in S$ and $y = (y_1, y_2, \ldots, y_n)^T$ is such that y_i is the coordinate of $\mathbf{x}$

corresponding to the vertex $i \in \bar{S}$.
If S is an independent set, then $x^T A_{G[S]} x = 0$ and, since

$$\lambda_1 \sum_{i \in S} x_i^2 = \sum_{ij \in \partial(S)} x_i y_j = x^T B y\,,$$

from (5), it follows that

$$\lambda_1 = 2\lambda_1 \sum_{i \in S} x_i^2 + y^T A_{G[\bar{S}]} y. \tag{6}$$

For any $i \in \bar{S}$, by Cauchy-Schwartz inequality, we have

$$\frac{\sum_{j \in N_G(i) \cap \bar{S}} y_j}{{d_i}'} \leq \sqrt{\frac{\sum_{j \in N_G(i) \cap \bar{S}} {y_j}^2}{{d_i}'}} \leq \sqrt{\frac{1 - \sum_{j \in S} x_j^2 - y_i^2}{{d_i}'}}, \tag{7}$$

where ${d_i}' = |N_G(i) \cap \bar{S}|$. Hence,

$$\sum_{j \in N_G(i) \cap \bar{S}} y_j \leq \sqrt{{d_i}'} \sqrt{1 - \sum_{j \in S} x_j^2 - y_i^2} \leq \sqrt{\Delta'} \sqrt{1 - \sum_{j \in S} x_j^2 - y_i^2} \tag{8}$$

where $\Delta' = \max\limits_{i \notin S} {d_i}'$. Then,

$$y^T A_{G[\bar{S}]} y = (\sum_{j \in N_{G[\bar{S}]}(1)} y_j) y_1 + \ldots + (\sum_{j \in N_{G[\bar{S}]}(n)} y_j) y_n \leq \sqrt{\Delta'} \sum_{i \in \bar{S}} \left(\sqrt{1 - \sum_{j \in S} x_j^2 - y_i^2} \right) y_i. \tag{9}$$

Now, we look for the maximum of the function

$$F(y_1, \ldots, y_n) = \sqrt{\Delta'} \sum_{i \in \bar{S}} \left(\sqrt{1 - \sum_{j \in S} x_j^2 - y_i^2} \right) y_i \tag{10}$$

under the constraint

$$\sum_{i=1}^{n} y_i^2 = 1 - \sum_{j \in S} x_j^2. \tag{11}$$

For this purpose we introduce the *Lagrangian* associated with constrained problem:

$$G(y_1, \ldots, y_n, \mu) = F(y_1, \ldots, y_n) - \mu \left(\sum_{i=1}^{n} y_i^2 - (1 - \sum_{j \in S} x_j^2) \right). \tag{12}$$

The stationary points of the function $G(y_1, \ldots, y_n, \mu)$ are the solutions of the following system of the equations:

$$\frac{\partial G}{\partial y_i} = \sqrt{\Delta'} \left(\frac{-y_i^2}{\sqrt{1 - \sum_{j \in S} x_j^2 - y_i^2}} + \sqrt{1 - \sum_{j \in S} x_j^2 - y_i^2} \right) - 2\mu y_i = 0, \text{ for } i = 1, \ldots, n, \tag{13}$$

$$\frac{\partial G}{\partial \mu} = \sum_{i=1}^{n} y_i^2 - \left(1 - \sum_{j \in S} x_j^2 \right) = 0. \tag{14}$$

From (13) we obtain

$$y_i^2 = \left(1 - \sum_{j \in S} x_j^2\right)\left(\frac{1}{2} \pm \frac{\mu}{2\sqrt{\mu^2 + \Delta'}}\right), \qquad \text{for } i = 1, \dots, n. \tag{15}$$

Let us first determine the entries y_i such that $y_i^2 = \left(1 - \sum_{j \in S} x_j^2\right)\left(\frac{1}{2} + \frac{\mu}{2\sqrt{\mu^2+\Delta'}}\right)$.

Assuming that there are p such entries y_i, with $0 \leq p \leq n$, it follows that

$$p\left(1 - \sum_{j \in S} x_j^2\right)\left(\frac{1}{2} + \frac{\mu}{2\sqrt{\mu^2 + \Delta'}}\right) \leq 1 - \sum_{j \in S} x_j^2 \;\Leftrightarrow\; p\left(\frac{1}{2} + \frac{\mu}{2\sqrt{\mu^2 + \Delta'}}\right) \leq 1 \tag{16}$$

and then $p \leq 2$. Otherwise, we get a contradiction.
Therefore, $p \in \{0, 1, 2\}$.

- If $p = 2$, then $\mu = 0$ and $n = 2$. Thus we get the stationary point of $G(y_1, \dots, y_n, \mu)$:

$$(y_1^*, y_2^*, \mu^*) = (\sqrt{\frac{1 - \sum_{j \in S} x_j^2}{2}}, \sqrt{\frac{1 - \sum_{j \in S} x_j^2}{2}}, 0).$$

- If $p = 1$, then

$$\left(1 - \sum_{j \in S} x_j^2\right)\left(\frac{1}{2} + \frac{\mu}{2\sqrt{\mu^2 + \Delta'}}\right) + (n-1)\left(1 - \sum_{j \in S} x_j^2\right)\left(\frac{1}{2} - \frac{\mu}{2\sqrt{\mu^2 + \Delta'}}\right) = 1 - \sum_{j \in S} x_j^2,$$

which is equivalent to

$$\frac{n-2}{2}\left(1 - \frac{\mu}{\sqrt{\Delta' + \mu^2}}\right) = 0.$$

Therefore, $n = 2$ or $\Delta' = 0$.

1. If $\Delta' = 0$, then G is bipartite, with S as one of the two color classes and we obtain the stationary points of the function $G(y_1, \dots, y_n, \mu)$:

$$(y_1^*, \dots, y_n^*, \mu^*) \in \{(\sqrt{1 - \sum_{j \in S} x_j^2}, 0, \dots, 0, \mu), \dots, (0, 0, \dots, \sqrt{1 - \sum_{j \in S} x_j^2}, \mu)\}$$

   where μ is arbitrary. But for any of these points $F(y_1^*, \dots, y_n^*) = 0$.
2. If $\Delta' \neq 0$, then $n = 2$ and we obtain the following two stationary points of the function $G(y_1, \dots, y_n, \mu)$:

$$\begin{aligned} (y_1^*, y_2^*, \mu^*) &= (\sqrt{1 - \sum_{j \in S} x_j^2}\sqrt{\frac{1}{2} - \frac{\mu}{2\sqrt{\mu^2 + \Delta'}}}, \sqrt{1 - \sum_{j \in S} x_j^2}\sqrt{\frac{1}{2} + \frac{\mu}{2\sqrt{\mu^2 + \Delta'}}}, \mu^*) \\ &\text{or} \\ &= (\sqrt{1 - \sum_{j \in S} x_j^2}\sqrt{\frac{1}{2} + \frac{\mu}{2\sqrt{\mu^2 + \Delta'}}}, \sqrt{1 - \sum_{j \in S} x_j^2}\sqrt{\frac{1}{2} - \frac{\mu}{2\sqrt{\mu^2 + \Delta'}}}, \mu^*) \end{aligned}$$

– If $p = 0$, then

$$n\left(1-\sum_{j\in S}x_j^2\right)\left(\frac{1}{2}-\frac{\mu}{2\sqrt{\mu^2+\Delta'}}\right)=1-\sum_{j\in S}x_j^2,$$

which is equivalent to $\mu^2 = \frac{(n-2)^2}{4(n-1)}\Delta'$. Therefore, we obtain the following stationary point of the function $G(y_1, \ldots, y_n, \mu)$:

$$(y_1^*, \ldots, y_n^*, \mu^*) = (\sqrt{\frac{1-\sum_{j\in S}x_j^2}{n}}, \ldots, \sqrt{\frac{1-\sum_{j\in S}x_j^2}{n}}, \frac{n-2}{2}\sqrt{\frac{\Delta'}{n-1}}).$$

According to the above analysis, we may say that the maximum of the function $F(y_1, \ldots, y_n)$, with $n \geq 2$, under the constraint (14), is attained at the point

$$(y_1^*, \ldots, y_n^*) = (\sqrt{\frac{1-\sum_{j\in S}x_j^2}{n}}, \ldots, \sqrt{\frac{1-\sum_{j\in S}x_j^2}{n}}) \tag{17}$$

and therefore

$$F(y_1, \ldots, y_n) \leq F(y_1^*, \ldots, y_n^*) = \sqrt{\Delta'}\sqrt{n-1}(1-\sum_{j\in S}x_j^2). \tag{18}$$

In case when $n = 1$ the graph in question is a star S_m and therefore bipartite with $\Delta' = 0$, which leads to $F(y_1, \ldots, y_n) = 0$, for any $(y_1, \ldots, y_n) \in \mathbb{R}^n$.

Now, taking into account (6) and (18), we obtain:

$$\lambda_1 \leq 2\lambda_1\sum_{j\in S}x_j^2 + \sqrt{\Delta'}\sqrt{n-1}(1-\sum_{j\in S}x_j^2) \tag{19}$$

$$\Updownarrow$$

$$\lambda_1 - \sqrt{\Delta'}\sqrt{n-1} \leq (2\lambda_1 - \sqrt{\Delta'}\sqrt{n-1})\sum_{j\in S}x_j^2. \tag{20}$$

As immediate consequence, we have the main result of this section.

Theorem 3 *Let G be a connected graph with index λ_1 and let $S \subset V(G)$ be an independent set. Let us assume also that Δ' is the maximum degree of the subgraph of G induced by $\bar{S} = V(G) \setminus S$, $n = |\bar{S}|$ and $2\lambda_1 - \sqrt{\Delta'}\sqrt{n-1} > 0$. Then*

$$\sum_{j\in S}x_j^2 \geq \frac{\lambda_1 - \sqrt{\Delta'}\sqrt{n-1}}{2\lambda_1 - \sqrt{\Delta'}\sqrt{n-1}}. \tag{21}$$

4 Characterization of some split graphs

Based on the results obtained in the previous section, we are in a position to introduce the following result.

Theorem 4 *Let G be a connected graph with index λ_1 and an independent set $S \subset V(G)$ such that $|\bar{S}| = n > 2$. Then G is a split graph such that $\sum_{k \in N_G(i) \cap S} d_k$ is constant for every $i \in \bar{S}$ if and only if*

$$\sum_{i \in S} x_i^2 = \frac{\lambda_1 - n + 1}{2\lambda_1 - n + 1} \tag{22}$$

with $\lambda_1 > n - 1$.

Proof Using the results obtained in the previous section, we may conclude the following.

1. The inequality (21) with $\lambda_1 > \frac{n-1}{2}$ holds as equality if and only if (19) with $\lambda_1 > \frac{n-1}{2}$ holds as equality.
2. The inequality (19) with $\lambda_1 > \frac{n-1}{2}$ holds as equality if and only if the principal eigenvector of G, $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, is such that $y = (y_1^*, \ldots, y_n^*)^T$, $y^T A_{G[\bar{S}]} y = F(y_1^*, \ldots, y_n^*)$ and $\lambda_1 > \frac{n-1}{2}$.
3. The equality $y^T A_{G[\bar{S}]} y = F(y_1^*, \ldots, y_n^*)$ with $y = (y_1^*, \ldots, y_n^*)^T$ and $\lambda_1 > \frac{n-1}{2}$ holds if and only if both inequalities in (7) hold as equality with $y = y^*$ and $\lambda_1 > \frac{n-1}{2}$.
4. Both inequalities in (7) with $y = y^*$ hold as equality and $\lambda_1 > \frac{n-1}{2}$ if and only if the entries y_j^* are all equal for $j \in N_G(i) \cap \bar{S}$ (as it is the case, by (17)) and $N_G(i) \cap \bar{S} = \bar{S} \setminus \{i\}$, for every $i \in \bar{S}$, i.e., each vertex in $\bar{S}$ is adjacent to all vertices in $\bar{S}$ and $\lambda_1 > \frac{n-1}{2}$.
5. The previous statement is equivalent to say that both inequalities in (7) with $y = y^*$ hold as equality and $\lambda_1 > \frac{n-1}{2}$ if and only if the entries y_j^* are all equal for $j \in N_G(i) \cap \bar{S}$ (the point defined in (17)), each vertex in $\bar{S}$ is adjacent to all vertices in $\bar{S}$, i.e., $\bar{S}$ induces a complete subgraph, and $\lambda_1 > \frac{n-1}{2}$.
6. The entries y_j^* are all equal for $j \in N_G(i) \cap \bar{S}$, each vertex in $\bar{S}$ is adjacent to all vertices in $\bar{S}$ and $\lambda_1 > \frac{n-1}{2}$ if and only if y^*, defined in (17), is the subvector of the principal eigenvector of G with entries corresponding to the vertices in $\bar{S}$ and $\bar{S}$ induces a complete subgraph (then $\Delta' = n - 1$ and, as will see later, $\lambda_1 > \frac{n-1}{2}$).
7. The vector $y = y^*$ is the subvector of the principal eigenvector of G with entries corresponding to the vertices in $\bar{S}$ and $\bar{S}$ induces a complete subgraph if and only if G is a split graph such that $\sum_{k \in N_G(i) \cap S} d_k$ is constant for every $i \in \bar{S}$. In fact, let us prove this equivalence.
   (a) Assume that $y = y^*$ (as defined in (17)) is the subvector of the principal eigenvector of G with entries corresponding to the vertices in $\bar{S}$ and $\bar{S}$ induces a complete subgraph. Therefore, G is a split graph. Furthermore, since $y_i^* = \sqrt{\dfrac{1 - \sum_{j \in S} x_j^2}{n}}$, for $i = 1, \ldots, n$, by the eigenvalue equations,

$\forall i \in S$

$$\lambda_1 x_i = d_i \sqrt{\frac{1 - \sum_{j\in S} x_j^2}{n}} \quad \Leftrightarrow \quad x_i = \frac{d_i}{\lambda_1}\sqrt{\frac{1 - \sum_{j\in S} x_j^2}{n}} \tag{23}$$

and $\forall i \in \bar{S}$

$$\lambda_1 \sqrt{\frac{1 - \sum_{j\in S} x_j^2}{n}} = (n-1)\sqrt{\frac{1 - \sum_{j\in S} x_j^2}{n}} + \sum_{k\in N_G(i)\cap S} \frac{d_k}{\lambda_1}\sqrt{\frac{1 - \sum_{j\in S} x_j^2}{n}} \tag{24}$$

$$\Updownarrow$$

$$\lambda_1 = n - 1 + \sum_{k\in N_G(i)\cap S} \frac{d_k}{\lambda_1}. \tag{25}$$

The equality (25) means that $\sum_{k\in N_G(i)\cap S} d_k$ is constant for every $i \in \bar{S}$ and also that $\lambda_1 > n-1$.

(b) Conversely, if G is a split graph such that $\sum_{k\in N_G(i)\cap S} d_k$ is constant for every $i \in \bar{S}$, setting $y = y^*$, the eigenvalue equations (23) and (24) hold, and then the vector $\mathbf{x}$ became defined as an eigenvector of A_G. Since its entries are all positive components, then $\mathbf{x}$ is the principal eigenvector of G associated to the eigenvalue λ_1 which is the positive root of the quadratic polynomial

$$p(\lambda) = \lambda^2 - (n-1)\lambda - \sum_{k\in N_G(i)\cap S} d_k, \tag{26}$$

where i is chosen arbitrarily from $\bar{S}$ and then $\lambda_1 > n-1$.

8. Finally, since (21) (with $\lambda_1 > \frac{n-1}{2}$) holds as equality if and only if G is a split graph (therefore, $\Delta' = n-1$ and $\lambda_1 > n-1$) such that $\sum_{k\in N_G(i)\cap S} d_k$ is constant for every $i \in \bar{S}$, the result follows.

Computing the positive root of the quadratic polynomial (26), it follows that $\forall i \in \bar{S}$

$$\lambda_1 = \frac{1}{2}\left(n - 1 + \sqrt{(n-1)^2 + 4\sum_{k\in N_G(i)\cap S} d_k}\right).$$

For the particular case of a complete split graph, denoting the independence number of G by $\alpha(G)$ and its clique number by $\omega(G)$, we may conclude the following corollary.

Corollary 3 *Let G be a graph such that $\alpha = \alpha(G)$ and $\omega = \omega(G) > 2$ and let $S \subset V(G)$ be a maximum independent set. Then G is a complete split graph if and only if*

$$\sum_{j\in S} x_j^2 = \frac{1}{2} - \frac{\omega - 1}{2\sqrt{(\omega-1)^2 + 4\omega\alpha}},$$

where the x_j's are the entries of the principal eigenvector of G corresponding to the vertices of S.

Proof Since the index of a complete split graph G is $\lambda_1 = \frac{\omega-1}{2} + \frac{1}{2}\sqrt{(\omega-1)^2 + 4\omega\alpha}$, applying Theorem 4, the result follows. $\square$

5 Numerical examples

The graph of order $\nu = 9$ depicted in the Figure 2 has as principal eigenvector:

$\mathbf{x}^T = [0.33610, 0.18607, 0.24307, 0.33610, 0.24307, 0.42779, 0.42779, 0.25191, 0.43797]$

and its spectrum is

$\sigma(G) = \{-3.11742, -1.65855, -1.61803, 0.00000, 0.00000, 0.00000, 0.61803, 1.17772, 4.59825\}$.

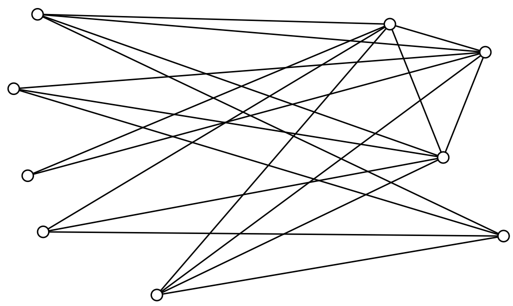

Fig. 2 A connected graph G with independence number $\alpha(G) = 5$

Applying Corollary 2, with $k = 0$, since $\underline{x} = 0.18607$ and $\bar{x} = 0.43797$ it follows

$$\begin{aligned}\alpha(G) &\leq \min\{\lfloor \frac{1}{2\underline{x}^2} \rfloor, \lfloor \nu - \frac{1}{2\bar{x}^2} \rfloor\} \\ &= \min\{\lfloor 14.44167 \rfloor, \lfloor 6.39336 \rfloor\} \\ &= 6.\end{aligned}$$

For $k = 1$ and $k = 2$, the obtained upper bounds for the order of k-regular induced subgraph is 6 and 7, respectively.

Considering the maximum independent set of G, S, since $n = \nu - \alpha(G) = 4$ and $\Delta' = 2$, then $4.59825 = \lambda_1 > \frac{\sqrt{\Delta'(n-1)}}{2} = \frac{\sqrt{2\times 3}}{2} = 1.22474$. Therefore, taking into account that the entries of the principal eigenvector, $\mathbf{x}$ of G, corresponding to the maximal independent set are the first 5 (below denote by $x_1, \ldots, x_5$), applying Theorem 4, we obtain

$$\begin{aligned}0.378715 = \sum_{j=1}^{5} x_j^2 \geq \frac{\lambda_1 - \sqrt{\Delta'(n-1)}}{2\lambda_1 - \sqrt{\Delta'(n-1)}} \\ = \frac{4.59825 - \sqrt{2 \times 3}}{2 \times 4.59825 - \sqrt{2 \times 3}} \\ = 0.318476.\end{aligned}$$

The graph H depicted in the Figure 3 has order $\nu = 6$ and principal eigenvector:

$$\mathbf{x}^T = [0.35877, 0.35877, 0.35877, 0.42099, 0.42099, 0.50931].$$

The spectrum of this graph is

$$\sigma(H) = \{-2.48361, -1.28282, 0, 0, 0, 3.76644\}.$$

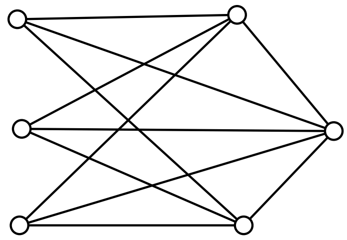

Fig. 3 A connected graph H with independence number $\alpha(H) = 3$

Applying Corollary 2, with $k = 0$, since $\underline{x} = 0.35877$ and $\bar{x} = 0.50931$ it follows

$$\begin{aligned}\alpha(H) &\leq \min\{\lfloor \frac{1}{2\underline{x}^2} \rfloor, \lfloor \nu - \frac{1}{2\bar{x}^2} \rfloor\}\\ &= \min\{\lfloor 3.88452 \rfloor, \lfloor 4.07245 \rfloor\}\\ &= 3.\end{aligned}$$

It is worth mentioning that, in this case, this upper bound on the stability number is better than the one obtained by Cvetković in [4] (see also [6, Theorem 3.10.1.]), where $\alpha(G) \leq \min\{\nu - \nu^+, \nu - \nu^-\}$, with ν^+ and ν^- denoting the number of positive and negative eigenvalues of G respectively. In this particular case, the bound obtained by Cvetković gives $\alpha(H) \leq 4$.

For $k = 1$ and $k = 2$, the obtained upper bounds for the order of k regular induced subgraph is 4, in both cases.

Considering the maximum independent set of G, S, since $n = \nu - \alpha(H) = 3$ and $\Delta' = 2$, then $3.76644 = \lambda_1 > \frac{\sqrt{\Delta'(n-1)}}{2} = \frac{\sqrt{2\times 2}}{2} = 1$. Therefore, taking into account that the entries of the principal eigenvector $\mathbf{x}$ of H, corresponding to the maximal independent set are the first 3 (below denote by x_1, x_2, x_3), applying Theorem 4, we obtain

$$\begin{aligned}0.38615 = \sum_{j=1}^{3} x_j^2 &\geq \frac{\lambda_1 - \sqrt{\Delta'(n-1)}}{2\lambda_1 - \sqrt{\Delta'(n-1)}}\\ &= 0.31926.\end{aligned}$$

References

1. D.M. Cardoso, M. Kamiński, V.V. Lozin, Maximum k-regular induced subgraphs, J. Comb. Optim., 14, 455-463 (2007)
2. D.M. Cardoso, P. Rowlinson, Spectral upper bounds for the order of a k-regular induced subgraph, Linear Algebra Appl., 433, 1031-1037 (2010)

3. S.M. Cioabă, A necessary and sufficient eigenvector condition for a connected graph to be bipartite, Electron. J. Linear Algebra, 20, 351-353 (2010)
4. D.M. Cvetković, Inequalities obtained on the basis of the spectrum of the graph, Studia Sci. Math. Hung., 8, 433-436 (1973)
5. D.M. Cvetković, M. Doob, H. Sachs, Spectra of Graphs, Theory and Applications, 3rd ed., Johan Ambrosius Barth Verlag, Heidelberg, (1995)
6. D.M. Cvetković, P. Rowlinson, S.K. Simić, An Introduction to the Theory of Graph Spectra, London Mathematical Society Student Texts, Cambridge University Press, Cambridge, (2010)